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Letter to the Editors

The Accuracy of Finite Fourier Transforms

1. Introduction

In my paper [1] I have pointed out that the calculation of finite Fourier integrals by the "fast Fourier transform" (F.F.T.) is sometimes very inaccurate, even when the number of integration points is not small. As an alternative I proposed to apply the F.F.T. not to the original function $f(t)$ but to another function $g(t)$, obtained by subtracting from $f(t)$ a linear trend so that $g(t)$ vanishes at both ends of the integration interval $[0, T]$. As a consequence, the F.F.T. applied to $g(t)$ gives the same result as the trapezoidal rule and therefore the calculation of

$$V = \int_0^T g(t)\, e^{-i\omega t}\, dt \tag{1}$$

by the F.F.T. is more accurate than that of

$$U = \int_0^T f(t)\, e^{-i\omega t}\, dt. \tag{2}$$

The correction to be added to V in order to obtain U is equal to the finite transform of the linear trend which is known analytically and therefore does not introduce any additional truncation errors. For any function $f(t)$ having a continuous second derivative and for the values $\omega_j = (2\pi j)/T$ considered when using the F.F.T. the proposed procedure, called the "trapezoidal F.F.T.," gives an error $E_1 = O(h^2)$ for the real part of (2), i.e. for the cosine-transform and an error $E_2 = O(h^3)$ for the imaginary part, i.e. for the sine-transform. An additional advantage of the proposed method is that it allows the performance of the Romberg process so that higher accuracy may be obtained without increasing essentially the computer time since the F.F.T. algorithm is based on successively doubling the number of integration points.

In order to illustrate the proposed method, I presented several tables comparing results for two types of oscillatory functions,

$$f(t) = t \cos \omega_0 t \tag{3}$$

and

$$f(t) = e^{-\alpha t} \sin \beta t, \tag{4}$$

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with results based on other methods, among them the classical Filon method. In two of the comparison tables the numbers based on the Filon procedure were in error, as pointed out by Kin-Chue Ng [2].

It is my purpose here to show that the usefulness of the trapezoidal F.F.T. method and the Romberg process based on it are not affected when the erroneous numbers are replaced by the correct ones. Also I want to comment here on other points raised by Kin-Chue Ng in his paper.

2. The Filon Method: Correct and Incorrect Results

The numbers given in paper [1, Tables I and II] as the results calculated by the Filon method are indeed incorrect, due to a flaw in the corresponding computer program. It is unfortunate to have a "bug" and to publish wrong results but as long as the checking of programs is a matter of experiment and not of mathematical proof, this may happen again and again. Moreover, from the numbers in Table I of this paper, it would be quite reasonable to think that the program giving the wrong numbers is correct, since e.g.

(a) for $\omega = 1$ and 1024 integration points, the result given by the incorrect program has *nine* correct digits, and

(b) the wrong numbers show improvement when the number of integration points is increased.

TABLE I

Values of $\phi_I(\omega) = \int_0^{2\pi} f(t) \sin \omega t \, dt, f(t) = t \cos t$

	Exact value	No. of integr. points	Incorrect (Filon)
1	-1.5707963268		
		64	$-1.5707\|615409$
		256	$-1.570796\|1907$
		1024	$-1.570796326\|7$
30	$-2.0967247966 \times 10^{-1}$		
		64	$1.782534660 \times 10^{-1}$
		256	$-2.0\|460924641 \times 10^{-1}$
		1024	$-2.096\|5120767 \times 10^{-1}$

As for the conclusions of paper [1], definitely they do not change for moderate and even large values of ω, as can be seen from Table II:

TABLE II

Values of $\phi_l(\omega) = \int_0^{2\pi} f(t) \sin \omega t \, dt, f(t) = t \cos 50t$

<table>
<tr><th></th><th>Exact value</th><th>No. of integr. points</th><th>Filon</th><th>Romberg</th></tr>
<tr><td>1</td><td>$2.51427983 \times 10^{-3}$</td><td></td><td></td><td></td></tr>
<tr><td></td><td></td><td>256</td><td>$2.|38303852 \times 10^{-3}$</td><td></td></tr>
<tr><td></td><td></td><td>1024</td><td>$2.51|390156 \times 10^{-3}$</td><td>$2.5142798|0 \times 10^{-3}$</td></tr>
<tr><td></td><td></td><td>2048</td><td>$2.5142|5652 \times 10^{-3}$</td><td></td></tr>
<tr><td>30</td><td>$1.178097245 \times 10^{-1}$</td><td></td><td></td><td></td></tr>
<tr><td></td><td></td><td>256</td><td>$1.1|21968687 \times 10^{-1}$</td><td></td></tr>
<tr><td></td><td></td><td>1024</td><td>$1.17|7921769 \times 10^{-1}$</td><td>$1.178097245| \times 10^{-1}$</td></tr>
<tr><td></td><td></td><td>2048</td><td>$1.1780|86350 \times 10^{-1}$</td><td></td></tr>
</table>

The result for $\omega = 1$ given by Filon's method even with 2048 points is not as accurate as that given by the trapezoidal F.F.T., while Romberg's process gives for 256, 512, and 1024 points three additional digits. The same pattern may be seen for $\omega = 30$.

3. The Equivalence of the Trapezoidal F.F.T. with the Usual Trapezoidal Rule

In the last part of his paper, Kin-Chue Ng gives a proof of the fact that the trapezoidal F.F.T. and the usual trapezoidal rule give the same result for the cosine transform of the function (3) when $T = 2\pi$ and ω_0 is an integer. In paper [1] and also in the present note we do not claim that the trapezoidal F.F.T. is identical with the usual trapezoidal rule for any function $f(t)$ but that they are *equivalent* i.e. that their truncation errors are of the same order: $O(h^2)$. This *equivalence holds for any function* $f(t)$ having a continuous second derivative and for any T and this has been shown in paper [1]. The difference between the truncation errors of the results given by the trapezoidal F.F.T. and the usual trapezoidal rule is precisely the truncation error of the trapezoidal rule for the linear trend, i.e. $O(h^2)$ and in this sense the two methods are equivalent. Kin-Chue Ng shows in his paper that in the particular case $T = 2\pi$ and $\omega_0 =$ integer, this last truncation error is practically zero, so that in this particular case the methods are not only equivalent but practically identical, but this does not contradict the equivalence in the general case.

One final remark related to formula (5) in [2]. It is much simpler to use this formula than to remove the linear trend as proposed in my paper [1], but there is a catch: As I have already explained, the removal of the linear trend gives for the sine transform a result which is more accurate than that given by the usual trapezoidal rule.

References

1. F. Abramovici, The accurate calculation of Fourier integrals by the fast Fourier transform technique, *J. Comput. Phys.* **11** (1973), 28.
2. Kin-Chue Ng, On the accuracy of numerical Fourier transforms, *J. Comput. Phys.* **16** (1974), 396.

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